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Year Five Pupils' Understanding of Generalised Arithmetic

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Abstract

This paper reports on the research on year five pupils' understanding of generalised arithmetic, which aimed to investigate the understanding of year five pupils' commutative property and the property of zero together with its correlation with their mathematical achievement. Data for the study were collected via paper and pencil assessment answers for two items. Findings showed moderate achievement for both of the tasks. The pupils' explanation illustrated their poor conceptual understanding of commutative property and the property of zero. However, this understanding is not correlated with their mathematical achievement in school. It shows that an outstanding student in school did not necessarily acquire conceptual understanding of commutative principle and the property of zero.

Keywords: *algebraic thinking, generalisation, properties of operations, primary school, early algebra*

Introduction

Algebraic thinking in primary school has received a great deal of attention in research among mathematics scholars. Researchers have found that infusing algebraic thinking in primary school will reduce the problems students face when they are exposed to formal algebra lessons in secondary school (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011). However, a question may arise regarding the way of infusing algebraic thinking in primary school. Does this mean teachers should start teaching algebra from primary school?

Infusing algebraic thinking in primary school definitely does not mean starting teaching algebra earlier in primary school (Carraher, Schliemann, & Schwartz, 2008). Algebraic thinking refers to emphasising properties of operations and making generalisations while working with arithmetic (Carpenter, Franke, & Levi, 2003; Slavit, 1999). Mathematics lessons often focus on computations, algorithms and correctness of solutions (Carpenter et al., 2003). Primary school pupils' exposure to properties of operations, generalisation, and working with patterns is generally very limited. Hence, secondary school students fail to see the connection between arithmetic and algebra (Herscovics & Linchevski, 1994). They tend to merely focus on formulae and algorithms when working with algebraic problems. Healthy classroom discussion during mathematics teaching and learning is essential to fill the cognitive gap between arithmetic and algebra.

Kaput (2008) theorized algebraic thinking into three main strands namely, generalised arithmetic, modelling and function. He defined generalised arithmetic as "the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalised arithmetic) and in quantitative reasoning" (p. 11), whereas modelling and function is associated with an ability to work on tasks involving equivalence, missing numbers, understanding of the equal sign and working with patterns.

Generalisation in arithmetic refers to "possessing the ability to identify generalisations concerning the fundamental properties of numbers" (Ralston, 2013, p. 23). Based on the literature on generalised arithmetic, this can be further classified into two important aspects, namely i) understanding the properties of operations and ii) understanding the properties and relationships of numbers. Properties of operations refer to the understanding of commutative, associative and distributive properties. Properties and relationships of numbers are associated with the understanding of odd and even numbers, zero and one. This indicates that the understanding of properties of numbers and operations is crucial, rather than an ability to find the correct solution.

According to the National Council of Teachers of Mathematics, "Analyzing the properties of the basic operations gives students opportunities to extend their thinking and to build a foundation for applying these understandings to other situations" (NCTM, 2000, p. 161). This statement can be interpreted as a habit of the mind of looking beyond a particular instant and ability to make a generalization, which helps to apply this understanding in different situations. Conceptual understanding of basic operation properties, especially understanding of commutativity, grows in power as the domain of numbers expands and as students consider all four operations (Schifter, Monk, Russell, & Bastable, 2008).

Russell, Schifter, and Bastable (2011) investigated how thte understanding the properties of operations and ability to generalise and justify enable pupils to develop fundamental ideas to learn formal algebra. In a teaching experiment involving grade 2 to grade 6 pupils, the authors found that good arithmetic instruction can lead to and strengthen fundamental understanding of the meaning of operations and change the ways of thinking. Their findings showed that students provided sensible answers based on their perspectives on numbers. When students were given a task to find the pairs of numbers that add to 25, they managed to identify 18 + 7 and 7+18 by explaining that the sum remains the same regardless of the order. It shows young children are capable of grasping the basic operations' properties. This understanding can be enhanced by teachers who can instil these elements in arithmetic classroom discussion.

Likewise, the findings of a study (Hunter, 2010) confirmed that understanding of commutative property could provide ample opportunities to generalise a situation and find a general solution, which acts as a crucial element of algebra. Hunter's study involved students aged from nine to eleven. The study aimed at investigating how understanding of commutative property helps to strengthen the ability to generate conjectures, justification and generalise. The findings showed the majority of the students' understanding of arithmetic properties deepened as a result of several classroom instruction sessions. It revealed exposure to commutative property enabled students to make sensible decisions by generating conjectures and justifying them. Appropriate classroom tasks and instructions lead them to develop generalisation.

Similarly, working with zero also provided opportunities for students to discover the role of zero as a powerful mathematical idea and arithmetic tool (Carpenter et al., 2003; Moss & McNab, 2011). Discovering the role of zero also helps to create generalisations. When the student is exposed to a number sentence, such as 5 + 0, they can explore the fact that adding zero to any number does not change the value of the number. Eventually, this knowledge can be further added by exposing the children to a number sentence such as 9 + 5 = 14 + 0. It is to provoke their understanding on the property of zero and also to familiarise them with operations and impress upon them that numbers can also appear after the equal sign (Carpenter et al., 2003). This could eradicate children's general misconception that the equal sign is always followed by an answer.

The discussion in the preceding sections showed the importance of knowing the commutative property and the role of zero in order to develop algebraic thinking skills earlier than secondary school. However, it is still questionable to what extent primary pupils in Malaysia understand the commutative property and the role of zero. There is no data to show Malaysian primary pupils' understanding on these two elements, which is crucial in generalised arithmetic.

Hence, we conducted this study to investigate year five pupils' understanding of generalised arithmetic. Even though samples from all over Malaysia would be ideal, because of time and cost constraints, this study was conducted only in the district of Malacca. The findings of this study may not be generalised to the Malaysian population. Yet, it definitely acts as an eye opener to many local researchers, educators and curriculum developers to look into ideas to infuse algebraic thinking in primary school. The following section discusses the methodology of the presented study.

Research Methodology

Objectives and Research Questions

As arithmetic generalisation is one of the major strands of algebraic thinking, the presented study aimed at investigating year five pupils' understanding of generalised arithmetic and its correlation with mathematical achievement in the district of Malacca in Malaysia. The objectives of this study are two-fold:

1. To explore the district of Malacca's year five pupils' understanding of generalised arithmetic.
2. To investigate if there is a relationship between year five pupils' understanding of generalised arithmetic and mathematics achievement.

In line with the objectives, this study aimed at answering the following research questions:

1. What is the year five pupils' understanding of generalised arithmetic?
2. Is there any significant relationship between year five pupils' understanding of generalised arithmetic and mathematical achievement?

Research Sample

The sample of the presented study are the year five pupils of national schools in the district of Malacca. The national schools were selected randomly from the school list provided by the Ministry of Education. All the year five pupils in the chosen schools were involved in the study. The total sample amounted to 720 (370 females (51.4%) and 350 males (48.6%)). Table 1 shows the sample's mid-year mathematics examination grades. 83.3% of the sample passed their school mid-year examination in mathematics.

Table 1. The sample's mid-year mathematics examination grades

Grade	Frequency	Percentage
A	118	16.4
B	156	21.7
C	203	28.2
D	122	16.9
E	120	16.7
Missing	1	0.10
Total	720	100.0

Instrument and Procedures

The data collected were based on the performance of the year five pupils in two tasks given as part of a larger study. The two tasks are attached in Appendix A. These tasks were adapted from Ralston (2013). The tasks were provided in both the English and Malay languages to prevent the language factor from influencing pupil performance. The tasks involved two sections. In the first section, the pupils were asked to choose the right answer. The second section required the pupils to write a short explanation for their choice of answer.

Data Analysis

The first section of the tasks was coded dichotomously. The correct answer was given 1 point, whereas an incorrect answer or *'don't know'* was given 0. The second section, which demands an explanation, was coded based on Ralston's (2013) coding rubrics (cf., Appendix B). The scoring rubric ranges from 0 to 2. Task 1 aimed at investigating the understanding of properties and relationships of numbers associated with zero, e.g., to test the pupils' understanding of the fact that multiplying a number by zero gives zero. Task 2 aimed at investigating the understanding of the commutative property. Descriptive statistics were used to answer the first research question. Inferential statistics were used to answer the second research question. The results are presented in the subsequent section.

Research Results

This section discusses the findings of the performance of the year five pupils based on the tasks given. Table 2 shows that the total number of pupils scored correctly and the respective percentages for both tasks. The year five pupils outperformed in Task 1 (53.5%) compared to Task 2 (45.4%). Overall, the year five pupils' performance is moderate in both tasks.

Table 2. Frequency and percentage of correct responses according to tasks.

Tasks	Frequency of correct response	Percentage
Task 1	385	53.5
Task 2	327	45.4

As discussed in the preceding section, each of the two tasks was also assessed based on the explanation provided for the selection of their answer. Table 3 shows the detailed results for the explanation provided in each task. No point was awarded for incorrect or no explanation provided. One point was given for the explanation which demonstrated partial understanding or did not provide sufficient reasoning for their choice of answers. Full points (2 points) were given for any explanation demonstrating understanding that multiplication of any number by zero always gives zero (for Task 1) and understanding that the order of numbers is always important in subtraction (for Task 2).

Task 1 explanation showed that 412 (57.2%) pupils gave incorrect or no explanation. 82 (11.4%) of the sample were awarded 1 point for their explanation which was correct but failed to provide sufficient reasoning. However, 226 (31.4%) pupils provided a correct explanation together with additional reasoning. These results showed 68.6% of the pupils lacked the understanding of the properties of zero, which means they failed to grasp the concept that the answer was always zero when any number was multiplied by zero. The understanding of this concept may look superficial for primary pupils. On the other hand, this concept actually plays an important role when working with formal algebra. This concept is widely used when solving a quadratic equation. Lack of understanding of the basic property of zero leads pupils to merely memorise the formula and follow the algorithm when they go to secondary schools.

Explanation for Task 2 (cf., Table 3) showed 554 (76.9%) pupils provided an incorrect answer or left it blank. 145 (20.1%) pupils managed to provide a correct or reasonable explanation but were unable to go beyond that, while only 21 (2.9%) pupils provided a correct answer with valid reasoning.

Table 3. Frequency and percentage of correct explanation according to tasks

Points	Task 1		Task 2	
	Frequency	Percentage	Frequency	Percentage
0	412	57.2	554	76.9
1	82	11.4	145	20.1
2	226	31.4	21	2.9

The pupils insufficiently demonstrated their understanding of commutative property for subtraction. The majority of the sample provided explanation it is true by quoting addition as an example. It shows they failed to comprehend the commutative law, which is only applicable for addition and multiplication but not for subtraction or division. Lack of commutative understanding might lead to poor foundation, which causes an inability to work with variables. The results show that only a handful of the sample are able to exhibit the understanding of commutative law with valid reasoning.

The relationship between the performance of the sample in generalised arithmetic tasks and their achievement in mathematics was tested with the use of the Pearson cross-product moment correlation. In order to cater to the Pearson-cross product moment correlation data requirements, scores for each task were calculated based on a 6-point scale. Points were given based on the performance in answering true or false and the explanation provided. Then scores for both tasks were summed to percentages. This fulfilled continuous data requirement. Table 4 summarises the relationships between the generalised arithmetic tasks and mathematical achievement. The data indicated that there was no correlation between the two variables, $r = 0.025$, p-value > 0.05. In other words, there is no correlation between the pupils' understanding of generalised arithmetic and classroom mathematics examination achievement. This could lead to the question: What does classroom examination actually test? Are the examination questions focused on testing the students' understanding of arithmetic properties or aimed only at correctness of the solutions given?

Table 4. The Pearson-product moment correlations of each task and mathematics achievement

	Tasks	Mathematics achievement
Tasks	1	
Mathematical achievement	0.025	1

Discussion

The study was conducted to investigate the district of Malacca year five pupils' understanding of generalised arithmetic and the correlation of this understanding with their mathematical achievement, particularly their knowledge of multiplication by zero with any number and the commutative law in subtraction. The results exhibit the year five pupils' performance in generalised arithmetic and the correlation with their mid-year mathematics examination performance.

Based on the responses given, it is evident that only half of the sample could answer the first section of each item correctly. They demonstrated an understanding that any number multiplied by zero gave zero and that the commutative principle is not applicable for subtraction. However, their explanation for their choice of answers provided further details about the understanding of generalised arithmetic. Only 31.4% and 2.9% of the sample demonstrated a conceptual understanding of multiplication of a number by zero and the commutative law. This is an early indication of the primary pupils' understanding of generalised arithmetic. The findings showed the sample did not acquire the properties of basic operations and understanding of zero. By right, the year five pupils should be able to provide a solid reason for their answer in Task 2. They had been taught arithmetic since year one and they had been working with basic arithmetic operations for at least four years of school education. Unfortunately, the results showed that about half of the sample were able to provide a correct answer but they were not able to explain it with a valid reason.

On the other hand, the findings for the second research question showed that their understanding of generalised arithmetic did not have any influence on their mathematical achievement. This is a point to ponder. An outstanding student in school does not necessarily possess a conceptual understanding of the commutative principle and the property of zero. It means that the school mathematical assessment does not really focus on conceptual understanding. It probably focuses on the correctness of a solution. It appears that it is possible for pupils to get the correct solution without knowing the underlying conceptual facts. As proven by the findings of this study, the pupils in the sample were unable to provide conceptual reasoning even though they were able to provide correct answers. Eventually, this will lead them to memorise algorithms and apply them to solve problems without fundamental understanding.

Conclusions

Educators and policy makers should begin to look into this issue. Professional development should be given to teachers to show how to build on students' emerging knowledge of numbers and operations to help them engage with the ideas of algebra (Schifter et al., 2008). Teachers and good classroom instruction play a crucial role in infusing algebraic thinking in primary schools. They also should know how to assess the emergence of algebraic thinking in class. Failure to identify the algebraic thinking elements by teachers may hinder the development of algebraic thinking in primary school.

References:

Carpenter, T.P., Franke, M.L., & Levi, L. (2003). Thinking mathematically: Integrating arithmetic and algebra in elementary school. Portsmouth, NH: Heinemann.

Carraher, D.W., Schliemann, A.D., & Schwartz, J.L. (2008). Early algebra is not the same as algebra early. In J.J. Kaput, D.W. Carraher, & M.L. Blanton, *Algebra in the early grades* (pp. 235–272). New York, NY: Taylor & Francis.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*(1), 59–78. doi: 10.1007/BF01284528

Hunter, J. (2010, April). Developing early algebraic reasoning through exploration of the commutative principle. In M. Joubert, & P. Andrews, *Developing early algebraic reasoning through exploration of the commutative principle*. Symposium conducted at the British Congress for Mathematics Education, Manchester, UK. Retrieved from http://www.bsrlm.org.uk/IPs/ip30–1/BSRLM-IP-30–1-14.pdf

Kaput, J.J. (2008). What is algebra? What is algebraic reasoning? In J.J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York, NY: Taylor and Francis.

Knuth, E.J., Alibali, M.W., McNeil, N.M., Weinberg, A., & Stephens, A.C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence & variable. *Zentralblatt für Didaktik der Mathematik*, *37*(1), 68–76. doi: 10.1007/BF02655899

McNeil, N.M., Fyfe, E.R., Petersen, L.A., Dunwiddie, A.E., & Brletic-Shipley, H. (2011). Benefits of practicing 4=2+2: Nontraditional problem formats facilitate children's understanding of mathematical equivalence. *Child Development*, *82*(5), 1620–1633. doi: 10.1111/j.1467–8624.2011.01622.x.

Moss, J., & McNab, S.L. (2011). An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and co-variation. In J. Cai, & E. Knuth (Eds.), *Advances in mathematics education* (pp. 277–301). New York, NY: Springer.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Ralston, N.C. (2013). The development and validation of a diagnostic assessment of algebraic thinking skills for students in the elementary grades (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (UMI No. 3588844)

Schifter, D., Monk, S., Russell, S.J., & Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the elementary grades? In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 413–447). Mahwah, NJ: Erlbaum.

Slavit, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics*, *37*(3), 251–274. doi: 10.1023/A:1003602322232

Appendix A

Task 1

12. Circle whether this is 'Always True' or 'Not Always True'.
Bulatkan sama ada yang berikut adalah 'Selalu Benar' atau 'Tidak Selalu Benar'.

$$a \times 0 = a$$

Circle:	Always True	Not Always True	Don't Know
Bulatkan:	*Selalu Benar*	*Tidak Selalu Benar*	*Tidak Tahu*

Explain.
Jelaskan.

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Task 2

15. Circle whether this is 'Always True' or 'Not Always True'.
Bulatkan 'Selalu Benar' atau 'Tidak Selalu Benar'.

$$a - b = b - a$$

Circle:	Always True	Not Always True	Don't Know
Bulatkan:	*Selalu Benar*	*Tidak Selalu Benar*	*Tidak Tahu*

Explain.
Jelaskan.

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Appendix B

Scoring Rubric for Task 1

Score Awarded	Scoring Rules for Generalizing ax0=a	Examples
2 points	Student's response is correct and shows effective reasoning. Student understands that a number multiplied by zero equals zero and that this is **always** the case. Student likely uses the words "anything", "zero", "always", "multiplying / timesing", etc. Student may provide an example but they provide additional reasoning.	• Usually when you have a number times 0 the answer is 0 • Anything times 0 will equal 0 • Let's say a is 4. 4x0=4 is incorrect. When you multiply anything with zero it always ends up being zero as an answer • a+0 would equal a but ax0 would equal 0 because anything times zero equals zero • If you have something times 0 it equals 0 no matter what it equals 0
1 point	Student's response is correct but the student either does not supply reasoning or the reasoning is undeveloped or incorrect. Student may provide only an example or not understand that this concept always works. Students may have partial understanding or supply part but not all of the answer.	• If a was 2 then 2x0=0 it will not work • Because it is multiplying and it would be 0 • Because like 1x0=0 it is always 0 because it can't be 1 unless you do 1x1 then it would be true • 0x0=0 • Because if it was 5x0 would be 0 that's how it works 5x0=0 • Because ax0 doesn't equal a. It equals 0. So ax0=0 not a.
0 points	Student's response is incoherent, incorrect, or unanswered. Student may simply repeat the problem, or not understand that the variables stand for numbers.	• Blank • I don't know • Because if a is negative then that answers wrong • Because if you times by a zero it will still be the same number • Because it depends if you are trying to use the commutative property • a could equal 8 and with the zero, it could equal to 8 again

Scoring Rubric for Task 2

Score Awarded	Scoring Rules for Generalizing a-b=b-a	Examples
2 points	Student's response is correct and shows effective reasoning. Student understands that the order of numbers does matter in subtraction sentences and that this is **always** the case. Student likely uses the words "same number", "always", "subtracting", "negative numbers" etc. Student may provide an example but they provide additional reasoning.	• When you subtract you can't do what you do in addition so 7-6=1 but you can't do 6-7 because it will be a negative answer • Because one is smaller than the other and you could get a negative number, a could = 5 and b could = 3, 5-3 is not equal to 3-5 • Since a is bigger than b, if b subtracted by a, the number will be negative but if and b are equal, the sum will always be 0
1 point	Student's response is correct but the student either does not supply reasoning or the reasoning is undeveloped or incorrect. Student may provide only an example or not understand that this concept always works. Students may have partial understanding or supply part but not all of the answer.	• One example is 9-8=1 and 8-9 you can't do because 8 is smaller than 9 • No because you would get a negative in one of them • If you subtract something it cannot go backward • It only works when a and b are the same number • If a=5 and b=3 5-3=2 but 3-5=-2 • Because you have to subtract the smaller number from the bigger • Because the small number can't take away the big number
0 points	Student's response is incoherent, incorrect, or unanswered. Student may simply repeat the problem, or not understand that the variables stand for numbers.	• Blank • I don't know • You never know what it will be plus I do not know what b and a is • You need to know the numbers first • What are the letters? • This is so hard • They are the same letters • They are not always equal • The letters can change